Functional Analysis

Bartosz Kwaśniewski

Faculty of Mathematics, University of Białystok

Lecture 10

Adjoint operators

math.uwb.edu.pl/∼zaf/kwasniewski/teaching

H – is a fixed Hilbert space.

Thm (Riesz-Fréchet) A map $f : H \to \mathbb{F}$ is a bounded linear functional $\iff$ there is $y \in H$ such that

$$f(x) = \langle x, y \rangle \text{ for every } x \in H,$$

and then $\|f\| = \|y\|$. Hence the map $H \ni y \longmapsto \langle \cdot, y \rangle \in H^*$ is an isometric antilinear isomorphism: $H \overset{anty}{\cong} H^*$.

Proof: If $f(x) := \langle x, y \rangle$, $x \in H$, for some $y \in H$, then $f \in H^*$ and $\|f\| = \|y\|$ (see the proof of **Prop** from Lecture 7). Hence $H \ni y \longmapsto \langle \cdot, y \rangle \in H^*$ is an isometry, which is antilinear, because the scalar product is antilinear in the second argument.

Let now $f \in H^*$ be arbitrary. We can assume that $f \neq 0$, as if $f \equiv 0$, then $f(x) = \langle x, 0 \rangle$ for $x \in H$. Since $M := \ker f \neq H$, we get $\{0\} \neq M^\perp \subseteq H$. In fact we claim that $\dim(M^\perp) = 1$. Indeed, if $y_1, y_2 \in M^\perp \setminus \{0\}$, then $f(y_1), f(y_2) \neq 0$ and for $\lambda := \frac{f(y_2)}{f(y_1)} \in \mathbb{F}$

$$f(\lambda y_1 - y_2) = \lambda f(y_1) - f(y_2) = f(y_2) - f(y_2) = 0.$$

Hence $\lambda y_1 - y_2 \in \ker f = M$. But $\lambda y_1 - y_2 \in M^\perp$ ($M^\perp$ is a linear space). Thus $y_2 = \lambda y_1$, as $M \cap M^\perp = \{0\}$. Hence $\dim(M^\perp) = 1$.

Take any $y_0 \in M^\perp$ such that $\|y_0\| = 1$. Then for $x \in H$ we have $P_{M^\perp} x = \langle x, y_0 \rangle y_0$ (as $M^\perp = \{\lambda y_0 : \lambda \in \mathbb{F}\}$) and therefore

$$f(x) = f(P_M x + P_{M^\perp} x) = f(P_M x) + f(P_{M^\perp} x) = 0 + f(\langle x, y_0 \rangle y_0)$$
$$= \langle x, y_0 \rangle f(y_0) = \langle x, \overline{f(y_0)} y_0 \rangle.$$

Hence putting $y := \overline{f(y_0)} y_0$ we get $f(x) = \langle x, y \rangle$, $x \in H$. ■

Cor. Let (Ω, Σ, μ) be a measure space. Every bounded linear functional $f : L^2(\mu) \to \mathbb{F}$ is of the form

$$f(x) = \int_\Omega x(t) y(t) \, d\mu, \qquad x \in L^2(\mu),$$

where $y \in L^2(\mu)$, and $\|f\| = \|y\|_2 = \left(\int_\Omega |y(t)|^2 \, d\mu \right)^{\frac{1}{2}}$.

Thm. If $T : H \to K$ is a bounded linear operator between two Hilbert spaces H and K, then there exists exactly one function $T^* : K \to H$ such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \qquad \text{for all } x \in H,\ y \in K. \tag{1}$$

Moreover, $T^* \in B(K, H)$, $\|T^*\| = \|T\|$ and $(T^*)^* = T$.

Def. T^* is called the (Hermitian) **adjoint** of the operator T

Proof: For fixed $y \in K$ the map $f(x) := \langle Tx, y \rangle$, $x \in H$, is a bounded functional on H. In particular,

$$|f(x)| = |\langle Tx, y \rangle| \leqslant \|Tx\| \cdot \|y\| \leqslant \|T\| \cdot \|y\| \cdot \|x\|,$$

whence $\|f\| \leqslant \|T\| \cdot \|y\|$. Hence by **Thm**. (Riesz-Fréchet) there is a unique vector in H, that we denote by $T^* y$, such that $f(x) = \langle x, T^* y \rangle$, $x \in H$, that is $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for $x \in H$. Moreover, $\|T^* y\| = \|f\| \leqslant \|T\| \cdot \|y\|$. This proves existence and uniqueness of $T^* : K \to H$ satisfying (1).

T^* is linear, because for $y_1,\ y_2 \in K,\ \lambda \in \mathbb{F}$ and $x \in H$

$$\begin{aligned}\langle x,\, T^*(\lambda y_1 + y_2)\rangle &= \langle Tx, \lambda y_1 + y_2\rangle = \overline{\lambda}\langle Tx, y_1\rangle + \langle Tx, y_2\rangle \\ &= \overline{\lambda}\langle x,\, T^* y_1\rangle + \langle x,\, T^* y_2\rangle \\ &= \langle x, \lambda T^* y_1\rangle + \langle x,\, T^* y_2\rangle = \langle x, \lambda T^* y_1 + T^* y_2\rangle.\end{aligned}$$

Hence $T^*(\lambda y_1 + y_2) = \lambda T^* y_1 + T^* y_2$. From the previously obtained inequality $\|T^* y\| \leqslant \|T\| \cdot \|y\|$ we get $\|T^*\| \leqslant \|T\|$. To show the opposite inequality, let us note that the situation is symmetric and we can swap T and T^*. More precisely,

$$\langle T^* x, y\rangle = \overline{\langle y,\, T^* x\rangle} = \overline{\langle Ty, x\rangle} = \langle x,\, Ty\rangle,$$

whence $(T^*)^* = T$ and in particular $\|T\| = \|(T^*)^*\| \leqslant \|T^*\|$. ■

Ex. If $H = \mathbb{F}^n$ and $K = \mathbb{F}^m$, then for $A = [a_{i,j}]_{i=1,j=1}^{m,n} \in B(H,K)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \implies A^* = \begin{pmatrix} \overline{a_{11}} & \overline{a_{21}} & \dots & \overline{a_{m1}} \\ \overline{a_{12}} & \overline{a_{22}} & \dots & \overline{a_{m2}} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \dots & \overline{a_{nm}} \end{pmatrix}$$

Prop. (properties of adjoint) $T, S \in B(H,K)$, $R \in B(K,L)$

1. (a) involution: $(T^*)^* = T$
2. (b) antylinearity: $(\alpha T + \beta S)^* = \overline{\alpha} T^* + \overline{\beta} S^*$, for $\alpha, \beta \in \mathbb{F}$
3. (c) antimultiplicativity: $(RT)^* = T^* R^*$
4. (d) C^*-equality: $\|T\|^2 = \|T^* T\|$.

Proof: Property (a) was proved in the previous theorem.

(b) $\langle(\alpha T + \beta S)x, y\rangle = \alpha\langle Tx, y\rangle + \beta\langle Sx, y\rangle = \alpha\langle x, T^* y\rangle + \beta\langle x, S^* y\rangle$

$$= \langle x, \overline{\alpha} T^*\rangle + \langle x, \overline{\beta} S^* y\rangle = \langle x, (\overline{\alpha} T^* + \overline{\beta} S^*)y\rangle.$$

(c) $\langle RTx, y\rangle = \langle Tx, R^* y\rangle = \langle x, T^* R^* y\rangle$.

(d) Note that $\|T^* T\| \leqslant \|T^*\| \cdot \|T\|$ (since the operator norm is submultiplicative) and since $*$ is an isometry, we get $\|T^* T\| \leqslant \|T\|^2$. On the other hand, for $h \in H$ we have

$$\|Th\|^2 = \langle Th, Th\rangle = \langle h, T^* Th\rangle \overset{\text{Schwartz}}{\leqslant} \|h\| \|T^* Th\| \leqslant \|T^* T\| \|h\|^2$$

Thus $\|T\|^2 \leqslant \|T^* T\|$ and concluding $\|T\|^2 = \|T^* T\|$. ∎

Lem. For $U : H \to K$ the following conditions are equivalent:

1. U is an isometry,
2. U preserves the inner product,
3. $U^*U = 1$.

Proof: (1)$\Longrightarrow$(2). If U is an isometry, then by the polarization formulas for $x, y \in H$ and e.g. for $\mathbb{F} = \mathbb{C}$

$$\begin{aligned}\langle Ux, Uy\rangle &= \tfrac{1}{4}\textstyle\sum_{k=0}^{3} i^k \|Ux + i^k Uy\|^2 = \tfrac{1}{4}\sum_{k=0}^{3} i^k \|U(x + i^k y)\|^2 \\ &= \tfrac{1}{4}\textstyle\sum_{k=0}^{3} i^k \|x + i^k y\|^2 = \langle x, y\rangle.\end{aligned}$$

(2)$\Longrightarrow$(3). For any $x, y \in H$ we have

$$\langle Ux, Uy\rangle = \langle x, y\rangle \;\Leftrightarrow\; \langle x, U^*Uy\rangle = \langle x, y\rangle \;\Leftrightarrow\; \langle x, U^*Uy - y\rangle = 0.$$

Hence $U^*Uy - y = 0$, that is $U^*Uy = y$. Equivalently, $U^*U = 1$.

(3)$\Longrightarrow$(1). For any $x \in H$ we have

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle = \langle x, x\rangle = \|x\|^2. \qquad \blacksquare$$

Def. $U : H \to K$ is **unitary** if it is an invertible isometry.

Cor. U is a unitary operator if and only if $U^*U = UU^* = 1$.

Proof: If $U^*U = UU^* = 1$, then U is an invertible isometry, where $U^* = U^{-1}$, and so U is unitary. If U is unitary, then $U^*U = 1$, because U is an isometry, and so $U^{-1} = 1U^{-1} = U^*UU^{-1} = U^*$, that is $U^*U = UU^* = 1$. ∎

Characterization of operators in algebraic terms

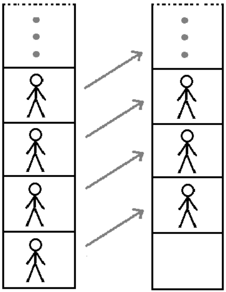
Relation	operator type
$T = T^*$	**self-adjoint**
$TT^* = T^*T$	**normal**
$P^2 = P,\ P = P^*$	**orthogonal projection**
$U^*U = 1$	**isometry**
$U^*U = UU^* = 1$	**unitary**

Rem. Unitary operator = „normal isometry".

Ex. (unilateral shift operator)

The operator $U : \ell^2 \to \ell^2$ given by

$$U(x(1), x(2), x(3), \dots) := (0, x(1), x(2), \dots)$$



is an isometry, but not a unitary, because $UH = \{x \in \ell^2 : x(1) = 0\} \neq H$. Moreover

$$U^*(x(1), x(2), x(3), \dots) = (x(2), x(3), \dots),$$

as

$$\langle Ux, y\rangle = \langle (0, x(1), x(2), x(3), \dots), (y(1), y(2), y(3), \dots)\rangle$$

$$= 0 \cdot \overline{y(1)} + x(1)\overline{y(2)} + x(2)\overline{y(3)} + \dots$$

$$= \langle (x(1), x(2), x(3), \dots), (y(2), y(3), \dots)\rangle = \langle x, U^* y\rangle.$$

In particular, $\ker U^* = \{x \in \ell^2 : x = (x(1), 0, 0, \dots)\} \neq \{0\}$ and

$$U^*U = 1, \qquad UU^* = P_{UH} = 1 - P_{\ker U^*} \neq 1.$$

Rem. $UU^* = P_{UH} = 1 - P_{\ker U^*}$ for any isometry U

Ex. (multiplication operators) Let $H = L^2(\mu)$, where (Ω, Σ, μ) is a measure space. For any $a \in L^\infty(\mu)$ the multiplication operator

$$(M_a x)(t) = a(t)x(t), \qquad x \in L^2(\mu),\ t \in \Omega,$$

is bounded and $\|M_a\| = \|a\|_\infty$ (see **Lecture 4**). It is easy to check that for any $a, b \in L^\infty(\mu)$ we have

$$(M_a)^* = M_{\bar{a}}, \qquad M_a M_b = M_{ab}.$$

Since multiplication of functions is commutative,**multiplication operators are normal**

$$M_a^* M_a = M_{\bar{a}a} = M_{|a|^2} = M_{a\bar{a}} = M_a M_a^*.$$

The above properties for the operator $M_a : L^2(\mu) \to L^2(\mu)$ depend only on the range of the function $a : \Omega \to \mathbb{C}$. Namely

$$M_a \text{ self-adjoint } \iff a \text{ is real valued } \mu\text{-a.e.}$$

$$M_a \text{ is a projection } \iff a \text{ attains only values } 0, 1\ \mu\text{-a.e.}$$

$$M_a \text{ is unitary } \iff a \text{ attains values in the unit circle } \mu\text{-a.e.}$$